7.2 Natural Logarithms

For any positive number a, the function value $f(x) = a^x$ is easy to define when x is an integer or rational number. When x is irrational, the meaning of a^x is not so clear. Similarly, the definition of the logarithm $\log_a x$, the inverse function of $f(x) = a^x$, is not completely obvious. In this section we use integral calculus to define the *natural logarithm* function, for which the number a is a particularly important value. This function allows us to define and analyze general exponential and logarithmic functions, $y = a^x$ and $y = \log_a x$.

Logarithms originally played important roles in arithmetic computations. Historically, considerable labor went into producing long tables of logarithms, correct to five, eight, or even more, decimal places of accuracy. Prior to the modern age of electronic calculators and computers, every engineer owned slide rules marked with logarithmic scales. Calculations with logarithms made possible the great seventeenth-century advances in offshore navigation and celestial mechanics. Today we know such calculations are done using calculators or computers, but the properties and numerous applications of logarithms are as important as ever.

Definition of the Natural Logarithm Function

One solid approach to defining and understanding logarithms begins with a study of the natural logarithm function defined as an integral through the Fundamental Theorem of Calculus. While this approach may seem indirect, it enables us to derive quickly the familiar properties of logarithmic and exponential functions. The functions we have studied so far were analyzed using the techniques of calculus, but here we do something more fundamental. We use calculus for the very definition of the logarithmic and exponential functions.

The natural logarithm of a positive number x, written as $\ln x$, is the value of an integral.

DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 7.9). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1. The function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0.$$

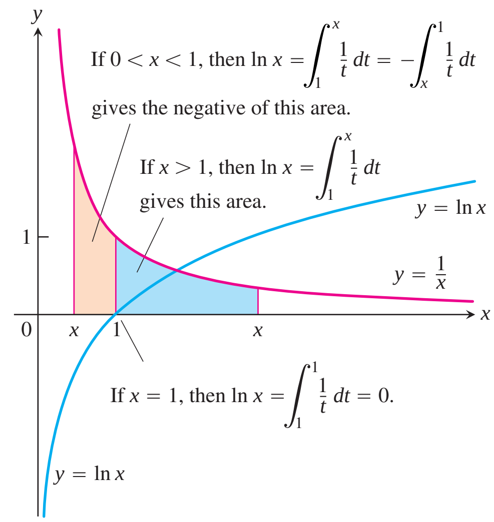


FIGURE 7.9 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

TABLE 7.1 Typical 2-place values of ln x

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

Notice that we show the graph of $y = 1/x$ in Figure 7.9 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x}\,dx,$$

with x meaning two different things. So we change the variable of integration to t.

By using rectangles to obtain finite approximations of the area under the graph of $y = 1/t$ and over the interval between $t = 1$ and $t = x$, as in Section 5.1, we can approximate the values of the function ln x. Several values are given in Table 7.1. There is an important number whose natural logarithm equals 1.

DEFINITION **The Number *e***

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

Geometrically, the number e corresponds to the point on the x-axis for which the area under the graph of $y = 1/t$ and above the interval $[1, e]$ is the exact area of the unit square. The area of the region shaded blue in Figure 7.9 is 1 sq unit when $x = e$.

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (Section 5.4),

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$. Notice that the derivative is always positive so the natural logarithm is an increasing function, hence it is one-to-one and invertible. Its inverse is studied in Section 7.3.

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \tag{1}$$

EXAMPLE 1 Derivatives of Natural Logarithms

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$

(b) Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2 + 3) = \frac{1}{x^2 + 3} \cdot \frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3} \cdot 2x = \frac{2x}{x^2 + 3}.$$

■

Notice the remarkable occurrence in Example 1a. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln ax$ for any positive number a:

$$\frac{d}{dx}\ln ax = \frac{1}{ax}\cdot\frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x}. \qquad (2)$$

Since they have the same derivative, the functions $y = \ln ax$ and $y = \ln x$ differ by a constant.

HISTORICAL BIOGRAPHY

John Napier
(1550–1617)

Properties of Logarithms

Logarithms were invented by John Napier and were the single most important improvement in arithmetic calculation before the modern electronic computer. What made them so useful is that the properties of logarithms enable multiplication of positive numbers by addition of their logarithms, division of positive numbers by subtraction of their logarithms, and exponentiation of a number by multiplying its logarithm by the exponent. We summarize these properties as a series of rules in Theorem 2. For the moment, we restrict the exponent r in Rule 4 to be a rational number; you will see why when we prove the rule.

THEOREM 2 Properties of Logarithms
For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln\frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln\frac{1}{x} = -\ln x$ — Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r\ln x$ — r rational

We illustrate how these rules apply.

EXAMPLE 2 Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$ — Product

(b) $\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$ — Quotient

(c) $\ln\frac{1}{8} = -\ln 8$ — Reciprocal

$= -\ln 2^3 = -3\ln 2$ — Power ■

EXAMPLE 3 Applying the Properties to Function Formulas

(a) $\ln 4 + \ln\sin x = \ln(4\sin x)$ — Product

(b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ — Quotient

(c) $\ln \sec x = \ln \frac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x+1} = \ln (x+1)^{1/3} = \frac{1}{3} \ln (x+1)$ Power ■

We now give the proof of Theorem 2. The steps in the proof are similar to those used in solving problems involving logarithms.

Proof that $\ln ax = \ln a + \ln x$ The argument is unusual—and elegant. It starts by observing that $\ln ax$ and $\ln x$ have the same derivative (Equation 2). According to Corollary 2 of the Mean Value Theorem, then, the functions must differ by a constant, which means that

$$\ln ax = \ln x + C$$

for some C.

Since this last equation holds for all positive values of x, it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln (a \cdot 1) &= \ln 1 + C \\ \ln a &= 0 + C \qquad \ln 1 = 0 \\ C &= \ln a. \end{aligned}$$

By substituting we conclude,

$$\ln ax = \ln a + \ln x.$$

Proof that $\ln x^r = r \ln x$ (assuming r rational) We use the same-derivative argument again. For all positive values of x,

$$\begin{aligned} \frac{d}{dx} \ln x^r &= \frac{1}{x^r} \frac{d}{dx}(x^r) \qquad \text{Eq. (1) with } u = x^r \\ &= \frac{1}{x^r} r x^{r-1} \\ &= r \cdot \frac{1}{x} = \frac{d}{dx}(r \ln x). \end{aligned}$$

Here is where we need r to be rational, at least for now. We have proved the Power Rule only for rational exponents.

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r \ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero, and we're done.

You are asked to prove Rule 2 in Exercise 84. Rule 3 is a special case of Rule 2, obtained by setting $a = 1$ and noting that $\ln 1 = 0$. So we have established all cases of Theorem 2. ■

We have not yet proved Rule 4 for r irrational; we will return to this case in Section 7.3. The rule does hold for all r, rational or irrational.

The Graph and Range of ln *x*

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down.

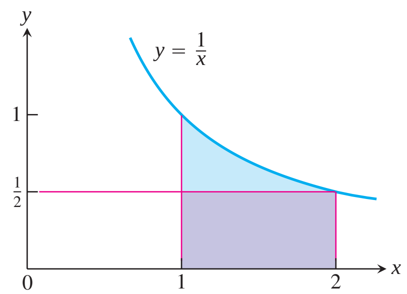


FIGURE 7.10 The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

We can estimate the value of ln 2 by considering the area under the graph of $y = 1/x$ and above the interval [1, 2]. In Figure 7.10 a rectangle of height 1/2 over the interval [1, 2] fits under the graph. Therefore the area under the graph, which is ln 2, is greater than the area, 1/2, of the rectangle. So $\ln 2 > 1/2$. Knowing this we have,

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and

$$\ln 2^{-n} = -n \ln 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}.$$

It follows that

$$\lim_{x\to\infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \ln x = -\infty.$$

We defined $\ln x$ for $x > 0$, so the domain of $\ln x$ is the set of positive real numbers. The above discussion and the Intermediate Value Theorem show that its range is the entire real line giving the graph of $y = \ln x$ shown in Figure 7.9.

The Integral $\int (1/u)\, du$

Equation (1) leads to the integral formula

$$\int \frac{1}{u}\, du = \ln u + C \tag{3}$$

when u is a positive differentiable function, but what if u is negative? If u is negative, then $-u$ is positive and

$$\int \frac{1}{u}\, du = \int \frac{1}{(-u)}\, d(-u) \qquad \text{Eq. (3) with } u \text{ replaced by } -u$$

$$= \ln(-u) + C. \tag{4}$$

We can combine Equations (3) and (4) into a single formula by noticing that in each case the expression on the right is $\ln |u| + C$. In Equation (3), $\ln u = \ln |u|$ because $u > 0$; in Equation (4), $\ln(-u) = \ln |u|$ because $u < 0$. Whether u is positive or negative, the integral of $(1/u)\, du$ is $\ln |u| + C$.

> If u is a differentiable function that is never zero,
>
> $$\int \frac{1}{u}\, du = \ln |u| + C. \tag{5}$$

Equation (5) applies anywhere on the domain of $1/u$, the points where $u \neq 0$.

We know that

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1 \text{ and rational}$$

Equation (5) explains what to do when n equals -1. Equation (5) says integrals of a certain *form* lead to logarithms. If $u = f(x)$, then $du = f'(x)\,dx$ and

$$\int \frac{1}{u}\,du = \int \frac{f'(x)}{f(x)}\,dx.$$

So Equation (5) gives

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

EXAMPLE 4 Applying Equation (5)

(a) $$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

$u = x^2 - 5, \quad du = 2x\,dx,$
$u(0) = -5, \quad u(2) = -1$

(b) $$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du$$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

$u = 3 + 2\sin\theta, \quad du = 2\cos\theta\,d\theta,$
$u(-\pi/2) = 1, \quad u(\pi/2) = 5$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Integrals of tan *x* and cot *x*

Equation (5) tells us at last how to integrate the tangent and cotangent functions. For the tangent function,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$

$u = \cos x > 0$ on $(-\pi/2, \pi/2)$,
$du = -\sin x\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C$$

Reciprocal Rule

$$= \ln|\sec x| + C.$$

For the cotangent,

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$$

$u = \sin x,$
$du = \cos x\,dx$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\int \tan u \, du = -\ln |\cos u| + C = \ln |\sec u| + C$$

$$\int \cot u \, du = \ln |\sin u| + C = -\ln |\csc x| + C$$

EXAMPLE 5

$$\int_0^{\pi/6} \tan 2x \, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u \, du$$

$$= \frac{1}{2} \ln |\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2} \ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 6 Using Logarithmic Differentiation

Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\ln y = \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}$$

$$= \ln ((x^2 + 1)(x + 3)^{1/2}) - \ln (x - 1) \qquad \text{Rule 2}$$

$$= \ln (x^2 + 1) + \ln (x + 3)^{1/2} - \ln (x - 1) \qquad \text{Rule 1}$$

$$= \ln (x^2 + 1) + \frac{1}{2} \ln (x + 3) - \ln (x - 1). \qquad \text{Rule 3}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

A direct computation in Example 6, using the Quotient and Product Rules, would be much longer.

EXERCISES 7.2

Using the Properties of Logarithms

1. Express the following logarithms in terms of ln 2 and ln 3.
   a. $\ln 0.75$ b. $\ln(4/9)$ c. $\ln(1/2)$
   d. $\ln\sqrt[3]{9}$ e. $\ln 3\sqrt{2}$ f. $\ln\sqrt{13.5}$
2. Express the following logarithms in terms of ln 5 and ln 7.
   a. $\ln(1/125)$ b. $\ln 9.8$ c. $\ln 7\sqrt{7}$
   d. $\ln 1225$ e. $\ln 0.056$
   f. $(\ln 35 + \ln(1/7))/(\ln 25)$

Use the properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. a. $\ln\sin\theta - \ln\left(\frac{\sin\theta}{5}\right)$ b. $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$
   c. $\frac{1}{2}\ln(4t^4) - \ln 2$
4. a. $\ln\sec\theta + \ln\cos\theta$ b. $\ln(8x + 4) - 2\ln 2$
   c. $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1)$

Derivatives of Logarithms

In Exercises 5–36, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = \ln 3x$
6. $y = \ln kx$, k constant
7. $y = \ln(t^2)$
8. $y = \ln(t^{3/2})$
9. $y = \ln\frac{3}{x}$
10. $y = \ln\frac{10}{x}$
11. $y = \ln(\theta + 1)$
12. $y = \ln(2\theta + 2)$
13. $y = \ln x^3$
14. $y = (\ln x)^3$
15. $y = t(\ln t)^2$
16. $y = t\sqrt{\ln t}$
17. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$
18. $y = \frac{x^3}{3}\ln x - \frac{x^3}{9}$
19. $y = \frac{\ln t}{t}$
20. $y = \frac{1 + \ln t}{t}$
21. $y = \frac{\ln x}{1 + \ln x}$
22. $y = \frac{x\ln x}{1 + \ln x}$
23. $y = \ln(\ln x)$
24. $y = \ln(\ln(\ln x))$
25. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$
26. $y = \ln(\sec\theta + \tan\theta)$
27. $y = \ln\frac{1}{x\sqrt{x + 1}}$
28. $y = \frac{1}{2}\ln\frac{1 + x}{1 - x}$
29. $y = \frac{1 + \ln t}{1 - \ln t}$
30. $y = \sqrt{\ln\sqrt{t}}$
31. $y = \ln(\sec(\ln\theta))$
32. $y = \ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$
33. $y = \ln\left(\frac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$
34. $y = \ln\sqrt{\frac{(x + 1)^5}{(x + 2)^{20}}}$
35. $y = \int_{x^2/2}^{x^2} \ln\sqrt{t}\,dt$
36. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt$

Integration

Evaluate the integrals in Exercises 37–54.

37. $\int_{-3}^{-2} \frac{dx}{x}$
38. $\int_{-1}^{0} \frac{3\,dx}{3x - 2}$
39. $\int \frac{2y\,dy}{y^2 - 25}$
40. $\int \frac{8r\,dr}{4r^2 - 5}$
41. $\int_0^{\pi} \frac{\sin t}{2 - \cos t}\,dt$
42. $\int_0^{\pi/3} \frac{4\sin\theta}{1 - 4\cos\theta}\,d\theta$
43. $\int_1^2 \frac{2\ln x}{x}\,dx$
44. $\int_2^4 \frac{dx}{x\ln x}$
45. $\int_2^4 \frac{dx}{x(\ln x)^2}$
46. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$
47. $\int \frac{3\sec^2 t}{6 + 3\tan t}\,dt$
48. $\int \frac{\sec y\tan y}{2 + \sec y}\,dy$
49. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$
50. $\int_{\pi/4}^{\pi/2} \cot t\,dt$
51. $\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta$
52. $\int_0^{\pi/12} 6\tan 3x\,dx$
53. $\int \frac{dx}{2\sqrt{x} + 2x}$
54. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

Logarithmic Differentiation

In Exercises 55–68, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

55. $y = \sqrt{x(x+1)}$

56. $y = \sqrt{(x^2+1)(x-1)^2}$

57. $y = \sqrt{\dfrac{t}{t+1}}$

58. $y = \sqrt{\dfrac{1}{t(t+1)}}$

59. $y = \sqrt{\theta + 3}\sin\theta$

60. $y = (\tan\theta)\sqrt{2\theta+1}$

61. $y = t(t+1)(t+2)$

62. $y = \dfrac{1}{t(t+1)(t+2)}$

63. $y = \dfrac{\theta+5}{\theta\cos\theta}$

64. $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

65. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

66. $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

67. $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$

68. $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Theory and Applications

69. Locate and identify the absolute extreme values of

a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

b. $\cos(\ln x)$ on $[1/2, 2]$.

70. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

71. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

72. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

73. The region in the first quadrant bounded by the coordinate axes, the line $y = 3$, and the curve $x = 2/\sqrt{y+1}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

74. The region between the curve $y = \sqrt{\cot x}$ and the x-axis from $x = \pi/6$ to $x = \pi/2$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

75. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

76. In Section 6.2, Exercise 6, we revolved about the y-axis the region between the curve $y = 9x/\sqrt{x^3+9}$ and the x-axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π. What volume do you get if you revolve the region about the x-axis instead? (See Section 6.2, Exercise 6, for a graph.)

77. Find the lengths of the following curves.

a. $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$

b. $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

78. Find a curve through the point (1, 0) whose length from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx.$$

T **79. a.** Find the centroid of the region between the curve $y = 1/x$ and the x-axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.

b. Sketch the region and show the centroid in your sketch.

80. a. Find the center of mass of a thin plate of constant density covering the region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$.

b. Find the center of mass if, instead of being constant, the density function is $\delta(x) = 4/\sqrt{x}$.

Solve the initial value problems in Exercises 81 and 82.

81. $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}, \quad y(1) = 3$

82. $\dfrac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1$

T **83. The linearization of $\ln(1+x)$ at $x = 0$** Instead of approximating $\ln x$ near $x = 1$, we approximate $\ln(1+x)$ near $x = 0$. We get a simpler formula this way.

a. Derive the linearization $\ln(1+x) \approx x$ at $x = 0$.

b. Estimate to five decimal places the error involved in replacing $\ln(1+x)$ by x on the interval [0, 0.1].

c. Graph $\ln(1+x)$ and x together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1+x)$ seem best? Least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

84. Use the same-derivative argument, as was done to prove Rules 1 and 4 of Theorem 2, to prove the Quotient Rule property of logarithms.

Grapher Explorations

85. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \le 10$. What is going on? Explain.

86. Graph $y = \ln|\sin x|$ in the window $0 \le x \le 22$, $-2 \le y \le 0$. Explain what you see. How could you change the formula to turn the arches upside down?

87. a. Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2$, 4, 8, 20, and 50 together for $0 \le x \le 23$.

b. Why do the curves flatten as a increases? (*Hint:* Find an a-dependent upper bound for $|y'|$.)

88. Does the graph of $y = \sqrt{x} - \ln x$, $x > 0$, have an inflection point? Try to answer the question **(a)** by graphing, **(b)** by using calculus.

7.3 The Exponential Function

Having developed the theory of the function $\ln x$, we introduce the exponential function $\exp x = e^x$ as the inverse of $\ln x$. We study its properties and compute its derivative and integral. Knowing its derivative, we prove the power rule to differentiate x^n when n is *any* real number, rational or irrational.

The Inverse of ln x and the Number e

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.11,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \ln^{-1} x = 0.$$

The function $\ln^{-1} x$ is also denoted by $\exp x$.

In Section 7.2 we defined the number e by the equation $\ln(e) = 1$, so $e = \ln^{-1}(1) = \exp(1)$. Although e is not a rational number, later in this section we see one way to express it as a limit. In Chapter 11, we will calculate its value with a computer to as many places of accuracy as we want with a different formula (Section 11.9, Example 6). To 15 places,

$$e = 2.718281828459045.$$

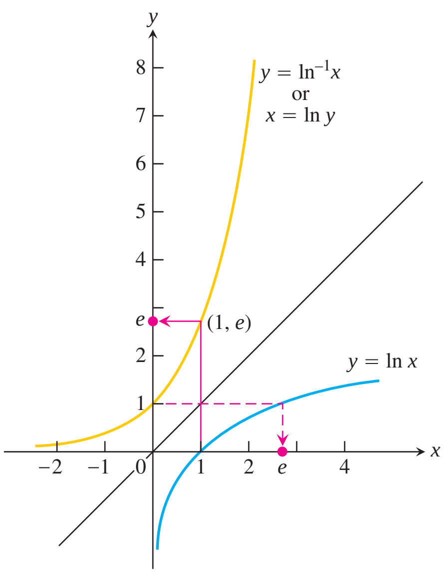


FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

The Function $y = e^x$

We can raise the number e to a rational power r in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take the logarithm, we find that

$$\ln e^r = r \ln e = r \cdot 1 = r.$$

Since $\ln x$ is one-to-one and $\ln(\ln^{-1} r) = r$, this equation tells us that

$$e^r = \ln^{-1} r = \exp r \qquad \text{for } r \text{ rational}. \tag{1}$$

We have not yet found a way to give an obvious meaning to e^x for x irrational. But $\ln^{-1} x$ has meaning for any x, rational or irrational. So Equation (1) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all x, so we use it to assign a value to e^x at every point where e^x had no previous definition.

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

DEFINITION The Natural Exponential Function
For every real number x, $e^x = \ln^{-1} x = \exp x$.

For the first time we have a precise meaning for an irrational exponent. Usually the exponential function is denoted by e^x rather than $\exp x$. Since $\ln x$ and e^x are inverses of one another, we have

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{2}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{3}$$

Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers that are not algebraic are called **transcendental**, like e and π. In 1873, Charles Hermite proved the transcendence of e in the sense that we describe. In 1882, C.L.F. Lindemann proved the transcendence of π.

Today, we call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

The domain of $\ln x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.

EXAMPLE 1 Using the Inverse Equations

(a) $\ln e^2 = 2$

(b) $\ln e^{-1} = -1$

(c) $\ln \sqrt{e} = \frac{1}{2}$

(d) $\ln e^{\sin x} = \sin x$

(e) $e^{\ln 2} = 2$

(f) $e^{\ln(x^2+1)} = x^2 + 1$

(g) $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way

(h) $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way ■

EXAMPLE 2 Solving for an Exponent

Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$e^{2k} = 10$$
$$\ln e^{2k} = \ln 10$$
$$2k = \ln 10 \qquad \text{Eq. (3)}$$
$$k = \frac{1}{2}\ln 10.$$ ■

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$.

HISTORICAL BIOGRAPHY

Siméon Denis Poisson
(1781–1840)

EXAMPLE 3 Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$ ■

We study the calculus of general exponential functions and their inverses in the next section. Here we need the definition in order to discuss the laws of exponents for e^x.

Laws of Exponents

Even though e^x is defined in a seemingly roundabout way as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra. Theorem 3 shows us that these laws are consequences of the definitions of $\ln x$ and e^x.

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Proof of Law 1 Let

$$y_1 = e^{x_1} \quad \text{and} \quad y_2 = e^{x_2}. \tag{4}$$

Then

$$\begin{aligned}
x_1 &= \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 && \text{Take logs of both sides of Eqs. (4).} \\
x_1 + x_2 &= \ln y_1 + \ln y_2 \\
&= \ln y_1 y_2 && \text{Product Rule for logarithms} \\
e^{x_1 + x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.} \\
&= y_1 y_2 && e^{\ln u} = u \\
&= e^{x_1} e^{x_2}.
\end{aligned}$$
■

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercise 78).

EXAMPLE 4 Applying the Exponent Laws

(a) $e^{x + \ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

(b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

(c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

(d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4 ■

Theorem 3 is also valid for a^x, the exponential function with base a. For example,

$$\begin{aligned} a^{x_1} \cdot a^{x_2} &= e^{x_1 \ln a} \cdot e^{x_2 \ln a} && \text{Definition of } a^x \\ &= e^{x_1 \ln a + x_2 \ln a} && \text{Law 1} \\ &= e^{(x_1 + x_2) \ln a} && \text{Factor } \ln a \\ &= a^{x_1 + x_2}. && \text{Definition of } a^x \end{aligned}$$

The Derivative and Integral of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero (Theorem 1). We calculate its derivative using Theorem 1 and our knowledge of the derivative of $\ln x$. Let

$$f(x) = \ln x \qquad \text{and} \qquad y = e^x = \ln^{-1} x = f^{-1}(x).$$

Then,

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx}(e^x) &= \frac{d}{dx} \ln^{-1} x \\ &= \frac{d}{dx} f^{-1}(x) \\ &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\ &= \frac{1}{f'(e^x)} && f^{-1}(x) = e^x \\ &= \frac{1}{\left(\frac{1}{e^x}\right)} && f'(z) = \tfrac{1}{z} \text{ with } z = e^x \\ &= e^x. \end{aligned}$$

That is, for $y = e^x$, we find that $dy/dx = e^x$ so the natural exponential function e^x is its own derivative. We will see in Section 7.5 that the only functions that behave this way are constant multiples of e^x. In summary,

$$\frac{d}{dx} e^x = e^x \qquad (5)$$

EXAMPLE 5 Differentiating an Exponential

$$\begin{aligned} \frac{d}{dx}(5e^x) &= 5 \frac{d}{dx} e^x \\ &= 5e^x \end{aligned}$$

■

The Chain Rule extends Equation (5) in the usual way to a more general form.

If u is any differentiable function of x, then

$$\frac{d}{dx}e^{u} = e^{u}\frac{du}{dx}. \tag{6}$$

EXAMPLE 6 Applying the Chain Rule with Exponentials

(a) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (6) with $u = -x$

(b) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cdot\cos x$ Eq. (6) with $u = \sin x$ ■

The integral equivalent of Equation (6) is

$$\int e^{u}\,du = e^{u} + C.$$

EXAMPLE 7 Integrating Exponentials

(a)
$$\begin{aligned}\int_0^{\ln 2} e^{3x}\,dx &= \int_0^{\ln 8} e^{u}\cdot\frac{1}{3}\,du \\ &= \frac{1}{3}\int_0^{\ln 8} e^{u}\,du \\ &= \frac{1}{3}e^{u}\Big]_0^{\ln 8} \\ &= \frac{1}{3}(8-1) = \frac{7}{3}\end{aligned}$$

$u = 3x,\quad \frac{1}{3}du = dx,\quad u(0) = 0,$
$u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

(b)
$$\begin{aligned}\int_0^{\pi/2} e^{\sin x}\cos x\,dx &= e^{\sin x}\Big]_0^{\pi/2} \\ &= e^{1} - e^{0} = e - 1\end{aligned}$$

Antiderivative from Example 6 ■

EXAMPLE 8 Solving an Initial Value Problem

Solve the initial value problem

$$e^{y}\frac{dy}{dx} = 2x, \qquad x > \sqrt{3}; \qquad y(2) = 0.$$

Solution We integrate both sides of the differential equation with respect to x to obtain

$$e^{y} = x^2 + C.$$

We use the initial condition $y(2) = 0$ to determine C:

$$\begin{aligned} C &= e^0 - (2)^2 \\ &= 1 - 4 = -3. \end{aligned}$$

This completes the formula for e^y:

$$e^y = x^2 - 3.$$

To find y, we take logarithms of both sides:

$$\begin{aligned} \ln e^y &= \ln(x^2 - 3) \\ y &= \ln(x^2 - 3). \end{aligned}$$

Notice that the solution is valid for $x > \sqrt{3}$.

Let's check the solution in the original equation.

$$\begin{aligned} e^y \frac{dy}{dx} &= e^y \frac{d}{dx} \ln(x^2 - 3) \\ &= e^y \frac{2x}{x^2 - 3} && \text{Derivative of } \ln(x^2-3) \\ &= e^{\ln(x^2-3)} \frac{2x}{x^2 - 3} && y = \ln(x^2-3) \\ &= (x^2 - 3)\frac{2x}{x^2 - 3} && e^{\ln y} = y \\ &= 2x. \end{aligned}$$

The solution checks. ■

The Number e Expressed as a Limit

We have defined the number e as the number for which $\ln e = 1$, or the value $\exp(1)$. We see that e is an important constant for the logarithmic and exponential functions, but what is its numerical value? The next theorem shows one way to calculate e as a limit.

THEOREM 4 The Number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) && \ln 1 = 0 \\ &= \lim_{x \to 0} \ln(1+x)^{1/x} = \ln\left[\lim_{x \to 0}(1+x)^{1/x}\right] && \text{ln is continuous.} \end{aligned}$$

Because $f'(1) = 1$, we have

$$\ln \left[\lim_{x \to 0} (1 + x)^{1/x} \right] = 1$$

Therefore,

$$\lim_{x \to 0} (1 + x)^{1/x} = e \qquad \text{ln } e = 1 \text{ and ln is one-to-one}$$ ■

By substituting $y = 1/x$, we can also express the limit in Theorem 4 as

$$e = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y. \tag{7}$$

At the beginning of the section we noted that $e = 2.718281828459045$ to 15 decimal places.

The Power Rule (General Form)

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n \ln x}$. Therefore, the n in the equation $\ln x^n = n \ln x$ no longer needs to be rational—it can be any number as long as $x > 0$:

$$\ln x^n = \ln (e^{n \ln x}) = n \ln x \qquad \ln e^u = u, \text{ any } u$$

Together, the law $a^x/a^y = a^{x-y}$ and the definition $x^n = e^{n \ln x}$ enable us to establish the Power Rule for differentiation in its final form. Differentiating x^n with respect to x gives

$$\begin{aligned}
\frac{d}{dx} x^n &= \frac{d}{dx} e^{n \ln x} && \text{Definition of } x^n,\ x > 0 \\
&= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) && \text{Chain Rule for } e^u \\
&= x^n \cdot \frac{n}{x} && \text{The definition again} \\
&= n x^{n-1}.
\end{aligned}$$

In short, as long as $x > 0$,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

The Chain Rule extends this equation to the Power Rule's general form.

Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}.$$

EXAMPLE 9 Using the Power Rule with Irrational Powers

(a) $\frac{d}{dx}x^{\sqrt{2}} = \sqrt{2}x^{\sqrt{2}-1} \qquad (x > 0)$

(b) $$\begin{aligned}\frac{d}{dx}(2 + \sin 3x)^{\pi} &= \pi(2 + \sin 3x)^{\pi-1}(\cos 3x) \cdot 3 \\ &= 3\pi(2 + \sin 3x)^{\pi-1}(\cos 3x).\end{aligned}$$

■

EXERCISES 7.3

Algebraic Calculations with the Exponential and Logarithm

Find simpler expressions for the quantities in Exercises 1–4.

1. a. $e^{\ln 7.2}$ **b.** $e^{-\ln x^2}$ **c.** $e^{\ln x - \ln y}$

2. a. $e^{\ln (x^2+y^2)}$ **b.** $e^{-\ln 0.3}$ **c.** $e^{\ln \pi x - \ln 2}$

3. a. $2 \ln \sqrt{e}$ **b.** $\ln (\ln e^e)$ **c.** $\ln (e^{-x^2-y^2})$

4. a. $\ln (e^{\sec \theta})$ **b.** $\ln (e^{(e^x)})$ **c.** $\ln (e^{2 \ln x})$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 5–10, solve for y in terms of t or x, as appropriate.

5. $\ln y = 2t + 4$ **6.** $\ln y = -t + 5$

7. $\ln (y - 40) = 5t$ **8.** $\ln (1 - 2y) = t$

9. $\ln (y - 1) - \ln 2 = x + \ln x$

10. $\ln (y^2 - 1) - \ln (y + 1) = \ln (\sin x)$

In Exercises 11 and 12, solve for k.

11. a. $e^{2k} = 4$ **b.** $100e^{10k} = 200$ **c.** $e^{k/1000} = a$

12. a. $e^{5k} = \frac{1}{4}$ **b.** $80e^k = 1$ **c.** $e^{(\ln 0.8)k} = 0.8$

In Exercises 13–16, solve for t.

13. a. $e^{-0.3t} = 27$ **b.** $e^{kt} = \frac{1}{2}$ **c.** $e^{(\ln 0.2)t} = 0.4$

14. a. $e^{-0.01t} = 1000$ **b.** $e^{kt} = \frac{1}{10}$ **c.** $e^{(\ln 2)t} = \frac{1}{2}$

15. $e^{\sqrt{t}} = x^2$ **16.** $e^{(x^2)}e^{(2x+1)} = e^t$

Derivatives

In Exercises 17–36, find the derivative of y with respect to x, t, or θ, as appropriate.

17. $y = e^{-5x}$ **18.** $y = e^{2x/3}$

19. $y = e^{5-7x}$ **20.** $y = e^{(4\sqrt{x}+x^2)}$

21. $y = xe^x - e^x$ **22.** $y = (1 + 2x)e^{-2x}$

23. $y = (x^2 - 2x + 2)e^x$ **24.** $y = (9x^2 - 6x + 2)e^{3x}$

25. $y = e^\theta(\sin \theta + \cos \theta)$ **26.** $y = \ln (3\theta e^{-\theta})$

27. $y = \cos (e^{-\theta^2})$ **28.** $y = \theta^3 e^{-2\theta} \cos 5\theta$

29. $y = \ln (3te^{-t})$ **30.** $y = \ln (2e^{-t} \sin t)$

31. $y = \ln \left(\frac{e^\theta}{1 + e^\theta} \right)$ **32.** $y = \ln \left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}} \right)$

33. $y = e^{(\cos t + \ln t)}$ **34.** $y = e^{\sin t}(\ln t^2 + 1)$

35. $y = \int_0^{\ln x} \sin e^t \, dt$ **36.** $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

In Exercises 37–40, find dy/dx.

37. $\ln y = e^y \sin x$ **38.** $\ln xy = e^{x+y}$

39. $e^{2x} = \sin (x + 3y)$ **40.** $\tan y = e^x + \ln x$

Integrals

Evaluate the integrals in Exercises 41–62.

41. $\int (e^{3x} + 5e^{-x}) \, dx$ **42.** $\int (2e^x - 3e^{-2x}) \, dx$

43. $\int_{\ln 2}^{\ln 3} e^x \, dx$ **44.** $\int_{-\ln 2}^{0} e^{-x} \, dx$

45. $\int 8e^{(x+1)} \, dx$ **46.** $\int 2e^{(2x-1)} \, dx$

47. $\int_{\ln 4}^{\ln 9} e^{x/2} \, dx$ **48.** $\int_0^{\ln 16} e^{x/4} \, dx$

49. $\int \frac{e^{\sqrt{r}}}{\sqrt{r}} \, dr$ **50.** $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} \, dr$

51. $\int 2t \, e^{-t^2} \, dt$ **52.** $\int t^3 e^{(t^4)} \, dt$

53. $\int \frac{e^{1/x}}{x^2} \, dx$ **54.** $\int \frac{e^{-1/x^2}}{x^3} \, dx$

55. $\int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta$ **56.** $\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta \, d\theta$

57. $\int e^{\sec \pi t} \sec \pi t \tan \pi t \, dt$

58. $\int e^{\csc (\pi + t)} \csc (\pi + t) \cot (\pi + t) \, dt$

59. $\displaystyle\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^{v}\cos e^{v}\,dv$

60. $\displaystyle\int_{0}^{\sqrt{\ln \pi}} 2x\,e^{x^2}\cos\left(e^{x^2}\right)dx$

61. $\displaystyle\int \frac{e^{r}}{1+e^{r}}\,dr$

62. $\displaystyle\int \frac{dx}{1+e^{x}}$

Initial Value Problems

Solve the initial value problems in Exercises 63–66.

63. $\dfrac{dy}{dt} = e^{t}\sin\left(e^{t}-2\right), \quad y(\ln 2) = 0$

64. $\dfrac{dy}{dt} = e^{-t}\sec^{2}\left(\pi e^{-t}\right), \quad y(\ln 4) = 2/\pi$

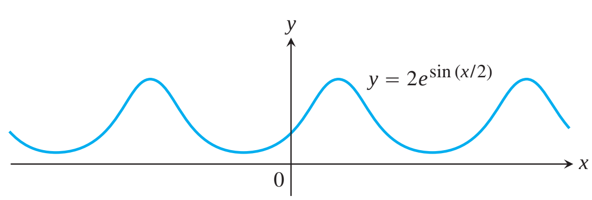
65. $\dfrac{d^{2}y}{dx^{2}} = 2e^{-x}, \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0$

66. $\dfrac{d^{2}y}{dt^{2}} = 1 - e^{2t}, \quad y(1) = -1 \quad \text{and} \quad y'(1) = 0$

Theory and Applications

67. Find the absolute maximum and minimum values of $f(x) = e^{x} - 2x$ on $[0, 1]$.

68. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



69. Find the absolute maximum value of $f(x) = x^{2}\ln(1/x)$ and say where it is assumed.

70. Graph $f(x) = (x-3)^{2}e^{x}$ and its first derivative together. Comment on the behavior of f in relation to the signs and values of f'. Identify significant points on the graphs with calculus, as necessary.

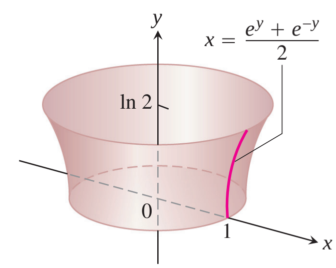
71. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^{x}$, and on the right by the line $x = \ln 3$.

72. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

73. Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_{0}^{1}\sqrt{1 + \frac{1}{4}e^{x}}\,dx.$$

74. Find the area of the surface generated by revolving the curve $x = (e^{y} + e^{-y})/2,\ 0 \le y \le \ln 2$, about the y-axis.



75. **a.** Show that $\int \ln x\,dx = x\ln x - x + C$.

b. Find the average value of $\ln x$ over $[1, e]$.

76. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

77. **The linearization of e^{x} at $x = 0$**

a. Derive the linear approximation $e^{x} \approx 1 + x$ at $x = 0$.

b. Estimate to five decimal places the magnitude of the error involved in replacing e^{x} by $1 + x$ on the interval $[0, 0.2]$.

c. Graph e^{x} and $1 + x$ together for $-2 \le x \le 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate e^{x}? Underestimate e^{x}?

78. **Laws of Exponents**

a. Starting with the equation $e^{x_1}e^{x_2} = e^{x_1 + x_2}$, derived in the text, show that $e^{-x} = 1/e^{x}$ for any real number x. Then show that $e^{x_1}/e^{x_2} = e^{x_1 - x_2}$ for any numbers x_1 and x_2.

b. Show that $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$ for any numbers x_1 and x_2.

79. **A decimal representation of e** Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$.

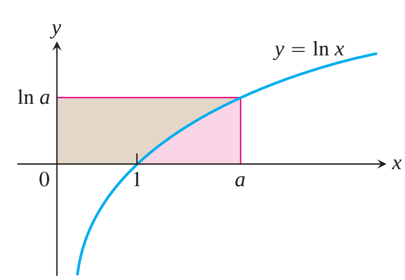
80. **The inverse relation between e^{x} and $\ln x$** Find out how good your calculator is at evaluating the composites

$$e^{\ln x} \quad \text{and} \quad \ln\left(e^{x}\right).$$

81. Show that for any number $a > 1$

$$\int_{1}^{a}\ln x\,dx + \int_{0}^{\ln a} e^{y}\,dy = a\ln a.$$

(See accompanying figure.)



82. **The geometric, logarithmic, and arithmetic mean inequality**

a. Show that the graph of e^{x} is concave up over every interval of x-values.

b. Show, by reference to the accompanying figure, that if $0 < a < b$ then

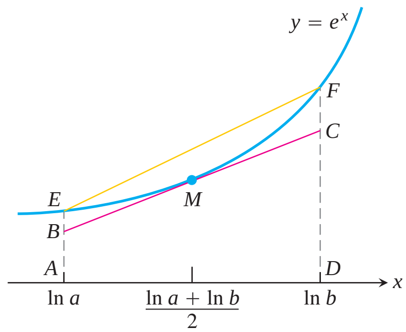
$$e^{(\ln a+\ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x \, dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a).$$

c. Use the inequality in part (b) to conclude that

$$\sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.

(For more about this inequality, see "The Geometric, Logarithmic, and Arithmetic Mean Inequality" by Frank Burk, *American Mathematical Monthly*, Vol. 94, No. 6, June–July 1987, pp. 527–528.)



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